Adaptive wavelet-based methods for solution of PDEs and signal analysis

By

Ratikanta Behera

Laboratoire Jean Kuntzmann (LJK)
Joseph Fourier University
Grenoble, France

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Outline

Academic details

Research Experience

- Post-Doc (January 2015 - Continue)
  Laboratory Jean kuntzmann (LJK),
  Joseph Fourier University,
  Grenoble, France.
- Post-Doc (December 2013 -December 2014)
  Center for Shock Wave-processing of Advanced Reactive Material (C-SWARM),
  University of Notre Dame,
  South Bend, Indiana, USA.

- Ph.D. in Mathematics, (July-2008 to Sept-2013),
  Indian Institute of Technology (IIT) Delhi, India.
  Thesis Title : Multilevel Adaptive Wavelet Methods for the Solution of PDEs,
  Under the supervision of Dr. Mani Mehra, Dept. of Mathematics, IIT Delhi, India.
- M.Phil. in Mathematics, (2006-2008),
  Utkal University, Bhubaneswar, Orissa, India.
- M.Sc. in Mathematics, (2004-2006),
  Utkal University, Bhubaneswar, Orissa, India.

Research Interests

- Wavelet Analysis and its Applications
  - Signal Processing
  - Numerical Analysis
  - Partial Differential Equation

Wavelets

- Mathematical modeling of problems ⇒ partial differential equations.
- Localized structures or sharp transitions
  - might occur intermittently anywhere in the computational domain
  - change their location and scales in space and time.
- Approximation of differential operators

Example :- Gradient, Jacobian, Surface divergence, Flux-divergence, Laplace-Beltrami operators

- Uniform grids is impractical, since high-resolution computations are required only in regions where sharp transitions occur.
- To solve these problems in a computationally efficient way, the computational grid should adapt dynamically in time to reflect local changes in the solution.
- Adaptive wavelet based method provide a simple, efficient and automatic way to adapt computational refinements to local demands of the solution.

✪ Fine resolution only in areas where it is needed and coarser resolution elsewhere.

✪ It required less number of coefficients to represent general functions and large data set accurately. This allows compression and efficient computations.

Multiresolution analysis on sphere

Decomposition of whole space into individual subspace, then study each individual subspaces.

Definition

MRA of the sphere provides a sequence of subspaces $V^j \subset L_2(S)$ with $j \geqslant 0$ and the sphere $S = \{p = (p_x, p_y, p_z) \in \mathbb{R}^3 : ||p|| = a\}$, where a is the radius of the sphere, such that

- $V^j \subset V^{j+1}$,
- $\bigcup_{j\geq 0} V^j$ is dense in $L_2(S)$,
- Each V^j has a Riesz basis of scaling functions $\{\phi_k^j | k \in K^j\}$.

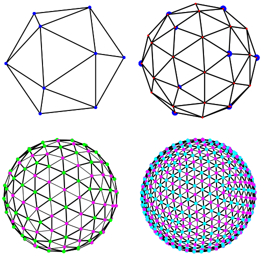
Define $W^j = \{\psi_m^j$ (wavelets) $| m \in M^j\}$ to be the complement of V^j in V^{j+1}, where $V^{j+1} = V^j \bigoplus W^j$,

$$\phi_k^j = \sum_{l \in K^{j+1}} h_{k,l}^j \phi_l^{j+1},$$

$$\psi_k^j = \sum_{l \in K^{j+1}} g_{k,l}^j \phi_l^{j+1}.$$

Wavelet on the sphere

- let $K^j = 10 \times 4^j + 2$ be the Icosahedral subdivision at subdivision level j.
- Let S be a triangulation of the sphere S $S^j = \{p_k^j \in S \mid p_k^j = p_{2k}^{j+1} k \in K^j\}$,
- Let $M^j = K^{j+1} \backslash K^j$ (added vertices when going from level j to $j+1$).



Fast Wavelet Transform

Analysis (j):

$$\forall m \in M^j : d_m^j = c_m^{j+1} - \sum_{k \in K_m} \tilde{s}_{k,m}^j c_k^j,$$

$$\forall m \in K^j : c_k^j = c_k^{j+1}$$

Synthesis (j):

$$\forall m \in M^j : c_m^{j+1} = d_m^j + \sum_{k \in K_m} \tilde{s}_{k,m}^j c_k^j,$$

$$\forall m \in K^j : c_k^{j+1} = c_k^j$$

Ex: For linear Interpolatory basis we can write, for $k \in K_m = \{v_1, v_2\}$ for analysis and synthesis:

$$d_m^j = c_m^{j+1} - \frac{1}{2}(c_{v_1}^{j+1} + c_{v_2}^{j+1}),$$
$$c_m^{j+1} = d_m^{j+1} + \frac{1}{2}(c_{v_1}^{j+1} + c_{v_2}^{j+1}),$$

Wavelet approximation

☞ Consider a function $u(p) \in L_2(S)$ can be express as

$$u(p) = \sum_{k \in K^0} c_k^0 \phi_k^0(p) + \sum_{j=0}^{\infty} \sum_{m \in M^j} d_m^j \psi_m^j(p)$$

composed this wavelets whose amplitudes are, above and below some prescribed ε threshold, such that $u(p) = u_{\geq}(p) + u_{<}(p)$, where

$$u_{\geq}(p) = \sum_{k \in K^0} c_k^{J_0} \phi_k^{J_0}(p) + \sum_{j=J_0}^{\infty} \sum_{\substack{m \in M^j \\ |d_m^j| \geq \varepsilon}} d_m^j \psi_m^j(p), \text{ and } u_{<}(p) = \sum_{j=J_0}^{\infty} \sum_{\substack{m \in M^j \\ |d_m^j| < \varepsilon}} d_m^j \psi_m^j(p),$$

$$\|u(p) - u_{\geq}(p)\|_\infty \leq c_1 \varepsilon,$$

and the number of significant coefficients $N(\varepsilon) = N$ depends on ε, s.t. $N(\varepsilon) \leq c_2 \varepsilon^{-n/d}$, where d is the order of interpolation, and n is the dimension of the problem,

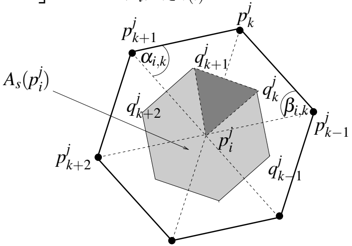
$$\|u(p) - u_{\geq}(p)\|_\infty \leq c_3 N(\varepsilon)^{-d/n}.$$

☞ We reconstruct $u_{\geq}(P)$ from $N(\varepsilon)$ of significant grid point.

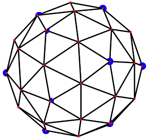
Approximation of the gradient operator

$$\nabla u(p_i^j) = \frac{1}{A_s(p_i^j)} \sum_{k \in N(i)} l_k \left[\frac{u(q_k^j) + u(q_{k+1}^j)}{2} \right] \mathbf{n}_k - \frac{u(p_i^j)}{A_s(p_i^j)} \sum_{k \in N(i)} l_k \mathbf{n}_k, \qquad (1)$$

- ☞ l_k is the length of the arc joining the triangle centroids q_k and q_{k+1},
- ☞ $\mathbf{n}_k$ is the outward unit normal vector to this arc at its midpoint,
- ☞ $N(i)$ is the set of nearest neighbour vertices of vertex p_i^j.
- ☞ $A_s(p_i^j)$ is the area of the one ring neighbourhood of p_i^j.



$$A_s(p_i^j) = \frac{1}{8} \sum_{k \in N(i)} (\cot \alpha_{i,k} + \cot \beta_{i,k}) \, ||p_k^j - p_i^j||^2,$$



☞ Ref: **R. Behera**, M. Mehra and N. K. R. Kevlahan, *Multilevel approximation of the gradient operator on an adaptive spherical geodesic grid*, *Adv. Comput. Math.*, 41 (2015) 663–689.

Approximation of the gradient operator

Gaussian function on the sphere

$$u(\theta,\phi) = 2\exp\left[-\frac{(\theta-\theta_0)^2+(\phi-\phi_0)^2}{L^2}\right],$$

where $\theta_0 = 0$ and $\phi_0 = 0$ and $L = 1/2\pi$.

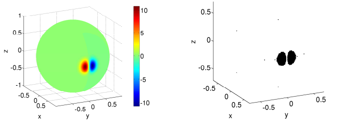


Figure : First component of the gradient operator (left) and the corresponding adaptive grid (right) with tolerance $\varepsilon = 10^{-3}$.

☞ Efficiency of the approximation of gradient operator is measured by : compression coefficient, $C = \frac{N(\varepsilon=0)}{N(\varepsilon\neq 0)}$.

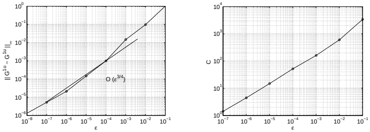


Figure : Control of error for the first component of the gradient operator in the L_∞ (left) and relation between compression coefficient (C) and ε (right)

☞ Ref: **R. Behera**, M. Mehra and N. K. R. Kevlahan, *Multilevel approximation of the gradient operator on an adaptive spherical geodesic grid*, *Adv. Comput. Math.*, 41 (2015) 663–689.

Barotropic vorticity equation

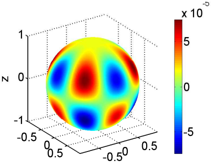
$$\frac{\partial \zeta}{\partial t} = -J(\psi, \zeta + f),$$

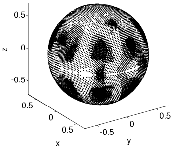
$$\zeta = \nabla^2 \psi.$$

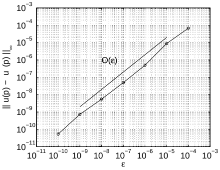
Here $\zeta(\theta,\phi,t)$ is the vorticity of the horizontal wind on the surface of the sphere. where $-\pi \leqslant \theta \leqslant \pi$ and $-\pi/2 \leqslant \phi \leqslant \pi/2$ are longitude and latitude respectively, ψ is the stream function,

$\zeta(\theta,\phi,t) = 2\omega \sin\phi - K \sin\phi \cos^R\phi (R^2 + 3R + 2)\cos R\theta$

with $\omega = K = 7.848 \times 10^{-6} s^{-1}$ and $R = 4$.







☞ Ref: **R. Behera** and M. Mehra, *Integration of barotropic vorticity equation on the sphere using adaptive wavelet collocation method*, *Appl. Math. Modelling,* 37 (2013) 5215–5226.

SynchroSqueezing Technique

☞ Developing a truly adaptive method for signal analysis is important for the understanding of many natural phenomena.

☞ Traditional signal analysis methods fall into two categories: linear and quadratic.

✎ Linear methods → efficient and easy to reconstruct but poor resolution.

✎ Quadratic methods → better resolution but suffer from higher computational cost, more difficult reconstruction process.

☞ In 1996, introduced in the context of analyzing auditory signals

☞ In 2011 an extensive mathematical analysis

☞ Synchrosqueezing by definition is a highly nonlinear operator along with special properties:

- sharpening the time-frequency representation from the wavelet transform.
- reconstructing the modes automatically from the reallocated coefficients.

Reference:

☞ I. Daubechies and S. Maes., *Wavelets in Medicine and Biology,* 3 (1996), 527–546.

☞ I. Daubechies, and J. Lu, and H.-T. Wu., *Appl. Comput. Harmon. Anal.,* 30 (2011), 243–261.

Multicomponent Signal

$$f(t) = \sum_{k=1}^{K} f_k(t), \quad \text{with } f_k(t) = A_k(t)e^{2i\pi\phi_k(t)}, \quad \text{for some finite } K,$$

- where $A_k(t)$ and $\phi_k(t)$ are time-varying amplitude and phase functions,
- $A_k(t) > 0$, $\phi_k'(t) > 0$ and $\phi_{k+1}'(t) > \phi_k'(t)$ for all t. ,
- $\phi_k'(t) \rightarrow$ Instantaneous Frequency (IF) and $A_k(t) \rightarrow$ Instantaneous Amplitude (IA).
- Consider $\widehat{\omega}(a,t) \rightarrow$ approximation of the IF of signal f at time t and frequency a.
- Let $W_f(a,t)$ is the wavelet transform of $f(t)$, then the IF estimates is defined as,

$$\widehat{\omega}_f(a,t) = \frac{1}{2\pi} I\left(\frac{\partial_t W_f(a,t)}{W_f(a,t)}\right), \quad \tilde{\omega}_f(a,t) = \frac{\partial_t W_f(a,t)}{2i\pi W_f(a,t)}$$

$$\left|\widehat{\omega}_f(a,t) - \phi_k'(t)\right| \leq ||\widetilde{\omega}_f(a,t) - \phi_k'(t)||.$$

★ Since wavelet time-frequency representation has a wide support spreading around the IF curve $(\phi'(t),t)$.

- The synchrosqueezing technique shifts the value of $W_f(a,t)$ from (a,t) to $(\widehat{\omega}_f(a,t),t)$, generating a sharpened TF representation $S_f(\omega,t)$ with a support concentrating around the curve $(\phi'(t),t)$.

Wavelet based synchrosqueezing transform

Let $\varepsilon > 0$ and $d > 0$, and $\mathscr{A}_{d,\varepsilon}$ is the set of multicomponent signals with modulation ε and separation d.

$$A_k \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}),\ \phi \in C^2(\mathbb{R}),\ \inf_{t\in\mathbb{R}} \phi_k'(t) > 0,\ \sup_{t\in\mathbb{R}} \phi_k'(t) < \infty,$$

$$A_k(t) > 0,\ |A_k'(t)| \le \varepsilon|\phi_k'(t)|,\ |\phi_k''(t)| \le \varepsilon|\phi_k'(t)|\ \ \forall t \in \mathbb{R}.$$

Further, the f_k are separated with resolution d, i.e., for all $k \in \{1, \ldots, K-1\}$ and all t

$$|\phi_{k+1}'(t) - \phi_k'(t)| \ge 2d(\phi_{k-1}'(t) + \phi_k'(t))$$

Definition of wavelet based synchrosqueezing transform

Let $h \in \mathbb{C}_c^\infty(\mathbb{R})$ be such that $\int_{\mathbb{R}} h(t) = 1$, and consider $\gamma, \delta > 0$, and $f \in \mathscr{A}_{d,\varepsilon}$, the Wavelet-based SynchroSqueezing Transform (WSST) of $f \in \mathscr{A}_{d,\varepsilon}$ with threshold γ and accuracy δ is defined by:

$$S_f^{\delta,\gamma}(\omega,t) = \int_{|W_f(a,t)|>\gamma} W_f(a,t)\frac{1}{\delta}h\left(\frac{\omega - \hat{\omega}_f(a,t)}{\delta}\right)a^{-3/2}da,$$

Motivation

Adaptive wavelet-based methods for solution of PDE

Adaptive wavelet-based methods for signal analysis

Future Plan

Publication

Analysis of adaptive wavelet-based method

Theorem -1 (R. Behera and S. Meignen)

Let $f(\tau) = Ae^{2i\pi\phi(\tau)}$ be a linear chirp, i.e. a wave with linear IF $\phi^{'}(\tau)$ and consider the estimate $\widehat{\omega}_f(a,t)$ computed using Morlet wavelet. If $|\frac{1}{a} - \phi^{'}(t)| \leq \varepsilon$ then

$$|\widehat{\omega}_f(a,t) - \phi^{'}(t)| \quad \leq \varepsilon \left| 1 - \frac{1}{1+\sigma^4 a^4 \phi^{''}(t)^2} \right|.$$

➥ Morlet wavelet is defined as $\psi(t) = \sigma^{-1} e^{-\pi \frac{t^2}{\sigma^2}} e^{2i\pi t}$, then we can write

$$\begin{aligned} W_f(a,t) &= \frac{1}{a}\int_{\mathbb{R}} f(\tau)\psi(\frac{\tau - t}{a})^* d\tau, \\ &= f(t) a \sigma^{-1} \left(\frac{1}{\sigma^2} - i a^2 \phi^{''}(t) \right)^{-\frac{1}{2}} e^{\left[\frac{-\pi\sigma^2 (1 - a\phi^{'}(t))^2}{1 - i\sigma^2 a^2 \phi^{''}(t)} \right]}. \\ \widehat{\omega}_f(a,t) &= \frac{1}{2\pi} I \left[2i\pi\phi^{'}(t) \right] + \left(\frac{\sigma^2 a \phi^{''}(t)(1 - a\phi^{'}(t))}{1 + \sigma^4 a^4 \phi^{''}(t)^2} (\sigma^2 a^2 \phi^{''}(t)) \right), \\ \Rightarrow |\widehat{\omega}_f(a,t) - \phi^{'}(t)| &= \left| \frac{\sigma^4 a^3 \phi^{''}(t)^2 (1 - a\phi^{'}(t))}{1 + \sigma^4 a^4 \phi^{''}(t)^2} \right| = \left| \frac{\sigma^4 a^4 \phi^{''}(t)^2 \left(\frac{1}{a} - \phi^{'}(t) \right)}{1 + \sigma^4 a^4 \phi^{''}(t)^2} \right|. \; \square \end{aligned}$$

Analysis of adaptive wavelet-based method

Theorem -2 (R. Behera and S. Meignen)

Let f be an hyperbolic chirp (i.e., $\frac{\phi''}{(\phi')^2}$ = constant) defined by $f(t) = (-it)^{2i\pi\alpha} = e^{\pi^2\alpha} e^{2i\pi\alpha\log(t)} = e^{\pi^2\alpha} e^{2i\pi\alpha\log(t)}$, with $\alpha < 1$, and consider the estimate $\widehat{\omega}_f(a,t)$ computed using the Cauchy wavelet with parameter 1, if $|\frac{1}{a} - \phi'(t)| \leq \varepsilon$ then

$$|\widehat{\omega}_f(a,t) - \phi'(t)| \quad \leq \left|\frac{\varepsilon}{1+\alpha^2}\right| + O(\varepsilon^2),$$

- ➥ For $\beta > \alpha$ we can write the Cauchy wavelet: $\psi_{2\pi\beta}(t) = \frac{1}{2\pi}\Gamma(1+2\pi\beta)(1-it)^{-(1+2\pi\beta)}$,
- ☞ Then $W_f(a,t) = ca^{2\pi\beta}(a-it)^{2\pi(i\alpha-\beta)}$, $c = \frac{1}{i\pi}\sinh(2\pi\alpha\pi)\Gamma(1+2i\pi\alpha)\Gamma(2\pi\beta-2i\pi\alpha)$.

$$\hat{\omega}_f(a,t) = \frac{\alpha}{t} + \alpha\left(\frac{t}{a^2+t^2} - \frac{1}{t}\right) + \frac{a\beta}{a^2+t^2},$$

$$\Rightarrow \left|\widehat{\omega}_f(a,t) - \phi'(t)\right| = \left|\frac{\frac{\beta}{a} - \frac{\alpha}{t}}{1+(\frac{t}{a})^2}\right|.$$

Consider $\beta = 1$ and $\phi'(t) = \frac{\alpha}{t}$, we get when $\alpha < 1$, $\left|\widehat{\omega}_f(a,t) - \phi'(t)\right| = \left|\frac{\frac{1}{a} - \phi'(t)}{1+(\frac{t}{a})^2}\right|$. □

Future Plan

Focus on Research

On signal processing

- Adaptive wavelet-based method for signal analysis
- Theoretical analysis of synchrosqueezing transforms
- Application on signal processing.

On solution of PDEs

- Synchrosqueezing technique for solution of PDEs.
- Curvelet optimized finite difference method for solutions PDEs.
- Simultaneous space-time adaptive wavelet methods for solution of PDEs.
- An adaptive wavelet method for the solution of integral equations on the sphere.

Future Plan

Focus on Teaching

Basic Courses

- Numerical Analysis
- Ordinary and/or Partial Differential Equation
- Linear Algebra

The following courses I would like to develop

- Introduction to Wavelets and its Applications
- Wavelets and Partial Differential Equation
- Advanced Computational Numerical Analysis of Partial Differential Equations
- Computational Aspects of Matrix Theory
- Introduction to Programming Language (i.e., C ,C ++, Fortran, Matlab)

Research Collaborators

- Dr. Sylvain Meignen,
  Joseph Fourier University,
  Grenoble, France.
- Dr. Mani Mehra,
  Indian Institute of Technology (IIT) Delhi,
  New Delhi, India.
- Prof. Nicholas Kevlahan,
  University of McMaster,
  Ontario, Canada.
- Prof. Samuel Paolucci,
  University of Notre Dame,
  South Bend, USA.

Publications

Published

- **R. Behera**, M. Mehra and N. K. R. Kevlahan, *Adv. Comput. Math.,* 41 (2015) 663–689, **Impact Factor:** 1.487
- **R. Behera** and M. Mehra, *Appl. Math. Modelling,* 37 (2013) 5215–5226, **Impact Factor:** 2.251
- **R. Behera** and M. Mehra, *Int. J. Wavelets Multiresolut. Inf. Process.,* 11 (2013) 1350019–1350032. **Impact Factor:** 0.694
- **R. Behera** and M. Mehra, *J. Multiscale Modelling,* 06 (1) (2015) 1450001–1430023.
- M. Mehra and **R. Behera,** *Indian Journal of Industrial and Applied Mathematics,* 4 (1) (2013) 1–14.

Publications

Submitted/Work in progress

Submitted

- R. Behera and M. Mehra, Approximation of the differential operators on an adaptive spherical geodesic grid using spherical wavelet,
- R. Behera, O. San, T. Grenga, K. Matous, and S. Paolucci, An adaptive Lagrangian-Eulerian wavelet algorithm for continuum mechanics,
- R. Behera and M. Mehra, An adaptive wavelet collocation method for solution of the convection dominated problem on the sphere.

Work in progress

- R. Behera and S. Meignen, Theoretical analysis of second-order synchrosqueezing transform.
- E. Martelli, S. Paolucci and R. Behera, Wavelet time integration scheme.

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- T. Oberlin, S. Meignen, V. Perrier, Second-order synchrosqueezing transform or invertible reassignment? towards ideal time-frequency representations, Signal Processing, IEEE Transactions on 63 (5) (2015) 1335–1344

Thank You!!!